Escuela Superior Politécnica del Litoral

PARES o NONES — Un break científico

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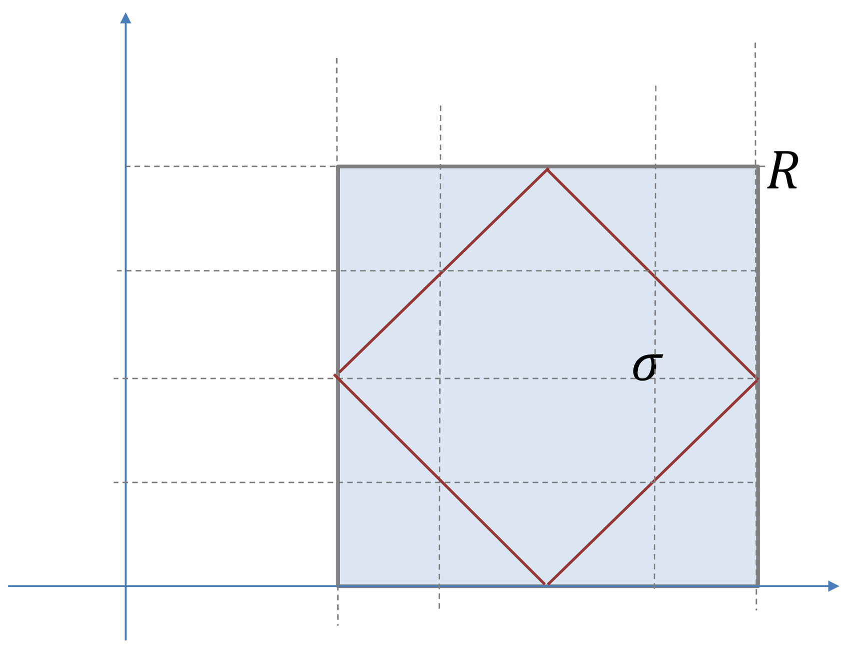
Bounded variation of functions defined on a convex and compact set in the plane

Problem

Bounded variation functions have been generalized since their initial presentation, so there are different definitions of bounded variation functions on rectangles in the plane. The matter addressed in the article is to define the variation of functions on a more general set of the plane.



General Objective

To define the variation of real functions of two variables, whose domain is a convex and compact set, and to study the vector space formed by functions with finite variation.

Proposal

Let σ be a compact and convex set in the plane, we will restrict ourselves to rectangle $R = [a,b] \times [c,d]$, which is the smallest rectangle that contains σ.

For each point $x \in [a,b]$ we define

$$\beta(x) := inf\,\{ y \in [c,d]:\ (x,y) \in \sigma \}$$
$$B(x) := sup\,\{ y \in [c,d]:\ (x,y) \in \sigma \}$$

and

$$V_{10}(f,\sigma) \coloneqq \sup\left\{\sum_{i=1}^{n}\left|f\big(x_i,\beta(x_i)\big) - f(x_{i-1},\beta(x_{i-1}))\right| : \xi = \{x_i\}_{i=1}^{n} \in \Lambda([a,b])\right\}$$

$$V_{01}(f,\sigma) \coloneqq \sup\left\{\sum_{i=1}^{m}\left|f\big(\alpha(y_i),(x_i)\big) - f(y_{i-1},\beta(y_i)\right| : \eta = \{y_i\}_{i=1}^{m} \in \Lambda([c,d])\right\}$$

$$V_{11}(f,\sigma) \coloneqq \sup\left\{\sum_{i=1}^{n}\left|f_v\big(x_{i-1},y_{j-1}\big) - f_v\big(x_i,y_{j-1}\big) + f_v\big(x_i,y_j\big) - f_v\big(x_{i-1},y_j\big)\right| : \xi \in \Lambda([a,b]), \eta = \{y_i\}_{i=1}^{m} \in \Lambda([c,d])\right\}$$

where

$$f_v(x,y) = \begin{cases} f(x,y), & (x,y) \in \sigma \\ f(x,\beta(x)) & (x,y) \in R - \sigma \text{ and } y < \beta(x) \\ f(x,B(x)) & (x,y) \in R - \sigma \text{ and } y > B(x). \end{cases}$$

Given this, the total variation of a function $f:\sigma \to \mathbb{R}$ is defined by $TV(f,\sigma) = V_{10}(f,\sigma) + V_{01}(f,\sigma) + V_{11}(f,\sigma)$

and the work is done on the space $BV(\sigma) := \{f:\sigma \to \mathbb{R} \mid TV(f,\sigma) < \infty\}$.

RESULTADOS

Properties.

1. If $f, g \in BV(\sigma)$, $\lambda, \gamma \in \mathbb{R}$, $\xi = \{x_i\}_{i=0}^{n}$ and $\eta = \{y_i\}_{i=0}^{m}$ are partitions of $[a,b]$ and $[e,d]$ $(e = \inf\{y \in [c,d]: \alpha(y) = a\})$ respectively, then for each $1 \le i \le n$ we will have
$TV(\lambda f + \gamma g, \sigma) \le |\lambda|\, TV(f,\sigma) + |\gamma|\, TV(g,\sigma)$.

2. $BV(\sigma)$, with the operations of addition and multiplication by a scalar, is a real vector space.

3. $TV(f,\sigma) = 0$ if and only if f is constant.

Theorem. Let $f:\sigma \to \mathbb{R}$ be a Lipschitz continuous function, with Lipschitz constant L, then $f \in BV(\sigma)$ and $TV(f,\sigma) \le 4L\ \delta(\sigma)$, where $\delta(\sigma)$ denotes the diameter of σ.

Theorem. $\|f\|_{BV(\sigma)} := \left|f\big(a,\beta(a)\big)\right| + TV(f,\sigma)$ defines a norm for $BV(\sigma)$.

Theorem. $BV(\sigma)$ is a Banach space.

We will say that the function $f:\sigma \to \mathbb{R}$ is non-decreasing if $f(\cdot,\beta(\cdot))$, $f(\alpha(\cdot),\cdot)$ are nondecreasing and $f(x_1,y_1) \le f(x_2,y_2)$ for any $a \le x_1 \le x_2 \le b$ and $c \le y_1 \le y_2 \le d$. This guarantees that the function f_v is non-decreasing.

Theorem. If a function $f:\sigma \to \mathbb{R}$ has a finite variation then there exist non-decreasing functions $g, h : \sigma \to \mathbb{R}$ such that $f = g - h$.

CONCLUSIONES

In this paper, we have presented a definition of variation for functions defined on sets that are more general than those defined in the plane, specifically on convex and compact sets. To do this, we extend the function defined to the smallest rectangle that contains the said set, thus leveraging the known properties of function variation over rectangles.

Subsequently, properties of this definition are presented, revealing that a set composed of functions defined on a convex and compact set, whose variation is bounded, is sufficiently rich as it contains all Lipschitz functions. Consequently, we obtain a space of functions equipped with a norm, with which it is demonstrated to be a Banach space. Finally, a Jordan-type decomposition theorem is obtained, meaning that a function in this space can be expressed as the difference of functions.